MATH-F410 – REP. DES GROUPES & APP. A LA PHYS.

Solutions des exercices 2022-2023

Séance 7 : Algèbres de Lie (I)

1. The adjoint representation

Let $\{X_a\}$ be a basis for the Lie algebra, in which the structure constants are $C^c{}_{ab}$ (so $[X_a, X_b] = C^c{}_{ab}X_c$). For generic vectors of the algebra $\xi = \xi^a X_a$ and $\eta = \eta^a X_a$, we have

$$\mathrm{ad}_\xi(\eta) = [\xi, \eta] = \xi^a \eta^b C^c{}_{ab} X_c = (\xi^a C^c{}_{ab})\, \eta^b X_c \ .$$

This is to be compared with $(\mathrm{ad}_\xi)^c{}_b \eta^b X_c$, which is the expression of the action of the previous map in terms of its matrix elements. We read

$$(\mathrm{ad}_\xi)^c{}_b = \xi^a C^c{}_{ab} \ .$$

If we set $\xi = X_a$, we get that matrix elements of the adjoint action of basis vectors are just the structure constants: $(\mathrm{ad}_{X_a})^c{}_b = C^c{}_{ab}$.

2. Ideals and quotient algebra

Recall that for any vector space V and subspace $W \subset V$, we can define the quotient vector space as $V/W = \{[v]\,/\,v \in V\}$, where $[v] = \{v + w\,/\,w \in W\}$ are the equivalence classes obtained from the equivalence relation in V $v \sim v' \Leftrightarrow v - v' \in W$. The set V/W inherits a natural structure of a vector space, where linear combinations are defined as $\alpha[v_1] + \beta[v_2] = [\alpha v_1 + \beta v_2]$ ($[0] \cong W$ is the neutral element, one can think of elements of V/W as subspaces "parallel" to W in V, for which we define the previous operation). Let us now go to Lie algebras. An ideal $I \subset L$ of the Lie algebra L is a subalgebra of L such that $[I, L] \subset I$ (meaning $[x, y] \in I$ for any $x \in I$, $y \in L$). In particular, being it a subalgebra, I is a linear subspace of L and we can apply the previous construction to obtain the quotient vector space L/I. We just have to check that it also inherits a natural Lie algebra structure. The only reasonable guess to define the bracket is:

$$\langle [x], [y] \rangle \equiv [[x, y]] \ .$$

Notation is a bit tricky here, so let us explain it clearly. On the left, $\langle , \rangle$ is the bracket defined on the quotient space L/I, whose elements are naturally equivalence classes $[x]$ for $x \in L$. On the right, we are first taking the bracket of x and y in L, $[x, y]$, and then computing the equivalence class of the resulting element, $[[x, y]]$, which is therefore an element of L/I. The properties of the bracket are inherited from those of L (linearity, antisymmetry), but there is a non-trivial check we must do. Our definition of the bracket must not depend on the representatives of the equivalence classes chosen, i.e., if $[x] = [x']$

and $[y] = [y']$, it must be $\langle [x], [y] \rangle = \langle [x'], [y'] \rangle$. This is a consequence of I being an ideal (this is why we need an ideal and not any subspace of L to construct the quotient algebra):

$$\begin{aligned}\langle [x'], [y'] \rangle &= [[x', y']] = [[x + w_1, y + w_2]] = [[x, y] + [x, w_2] + [w_1, y] + [w_1, w_2]] \\ &= [[x, y]] = \langle [x], [y] \rangle \ .\end{aligned}$$

The non-trivial step is the one going from the end of the first to the second line. w_1 and w_2 are elements of the ideal I (recall that any two representatives of an equivalence class in L/I, $[x'] = [x]$, are related as $x' = x + w$ for $w \in I$). Any commutators $[w_1, .]$ or $[w_2, .]$ are elements of I because of the definition of an ideal, and then they can be discarded for the computation of equivalence classes. This proves that the bracket is well-defined on L/I.

3. Invariant product of an algebra and adjoint action

Recall the adjoint action of the *group* G on its Lie algebra L: $\mathrm{Ad}_g(x) = gxg^{-1}$ for $g \in G$, $x \in L$. The statement of the question is just saying that we assume the existence of an inner product in L which is invariant under the adjoint action of the group: $g(\mathrm{Ad}_g(x), \mathrm{Ad}_g(y)) = g(x, y)$ (notation is a bit unfortunate here, g in the subscripts of Ad are group elements, while $g(x, y)$ denotes the Lie algebra inner product between x and y). We can consider an element of the group e^{tz} for t a real parameter in some neighborhood of the identity, and $z \in L$. Then, taking derivatives at $t = 0$ on both sides of the expression $g(x, y) = g(\mathrm{Ad}_{e^{tz}}(x), \mathrm{Ad}_{e^{tz}}(y))$ (and using bilinearity of the inner product, so that the usual product rule for derivatives applies):

$$\begin{aligned}0 &= \frac{d}{dt} g(x, y)\bigg|_{t=0} = \frac{d}{dt} g(e^{tz} x e^{-tz}, e^{tz} y e^{-tz})\bigg|_{t=0} \\ &= g\left(zx - xz, y\right) + g\left(x, zy - yz\right) \\ &= g\left([z, x], y\right) + g\left(x, [z, y]\right) \ .\end{aligned}$$

The result presented in the question then follows immediately.

4. Invariant product and structure constants

This is immediate writing the commutators of basis elements in terms of structure constants:

$$g([X_a, X_b], X_c) = g(X_a, [X_b, X_c]) \Leftrightarrow C^d{}_{ab} g_{dc} = C^d{}_{bc} g_{ad} \ .$$

5. Some examples of invariant forms

Section a)

Recall that $\mathfrak{u}(2) = \{X \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \,/\, X^\dagger = -X\}$, $\mathfrak{u}(1) = \{X \in \mathbb{C} \,/\, X^\star = -X\} = \{ia \,/\, a \in \mathbb{R}\}$, and $\mathfrak{su}(2) = \{X \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \,/\, X^\dagger = -X \,\&\, \mathrm{Tr}(X) = 0\}$. These are all clearly (real) vector spaces, and it is easy to provide bases for them:

$$\begin{aligned} \mathfrak{u}(2) &= \mathrm{span}_{\mathbb{R}} \{i\mathbf{1}_{2\times 2}, i\sigma_1, i\sigma_2, i\sigma_3\} \ , \\ \mathfrak{su}(2) &= \mathrm{span}_{\mathbb{R}} \{i\sigma_1, i\sigma_2, i\sigma_3\} \ , \\ \mathfrak{u}(1) &= \mathrm{span}_{\mathbb{R}} \{i\} \ , \end{aligned}$$

where σ_i are the Pauli matrices (which recall are Hermitian and traceless). It is clear then that we can write $\mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2)$ if we identify the $\mathfrak{u}(1)$ on the right as $\mathfrak{u}(1) \cong \mathrm{span}_{\mathbb{R}} \{i\mathbf{1}_{2\times 2}\}$ (clearly isomorphic to the above definition).

Section b)

Two elements $x, y \in \mathfrak{u}(1)$ can just be written as purely imaginary numbers, so $x = ia$, $y = ib$ for some $a, b \in \mathbb{R}$. Then:

- For the product α, it is immediate to compute $\alpha(ia, ib) = (ia)(ib) = -ab$.
- For the Killing product, notice $\mathrm{ad}_x(y) = [x, y] = 0$ for any $x, y \in \mathfrak{u}(1)$ because the algebra is Abelian. Thus, $\mathrm{ad}_x = 0$ for all $x \in \mathfrak{u}(1)$ and $k(ia, ib) = 0$.

Section c)

This is going to be more involved because the algebra is non-Abelian, so in particular the Killing product is non-trivial. For future computations, it is important to recall some well-known results about Pauli matrices:

$$\sigma_i \sigma_j = \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\sigma_k \ , \qquad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \ .$$

In particular, taking as a basis for $\mathfrak{u}(2)$ the set described in Section a),

$$\{X_0 = i\mathbf{1}_{2\times 2}, X_1 = i\sigma_1, X_2 = i\sigma_2, X_3 = i\sigma_3\} \ ,$$

it is immediate to check that non-zero structure constants in this basis are ${C^i}_{jk} = -2\epsilon_{ijk}$ (latin indices run over $1, 2, 3$, the indices of Pauli matrices). The adjoint actions of general elements $x, y \in \mathfrak{u}(2)$ have as matrix elements (see question 1) ${(\mathrm{ad}_x)^i}_j = -2x^k\epsilon_{ikj} = 2x^k\epsilon_{ijk}$ and ${(\mathrm{ad}_y)^i}_j = 2y^k\epsilon_{ijk}$.[1] This is enough to compute the products:

- For the product α, we get

  $$\alpha(x, y) = \mathrm{Tr}\left((ix^0\mathbf{1} + ix^j\sigma_j)(iy^0\mathbf{1} + iy^k\sigma_k)\right) = -2x^0y^0 - 2x^iy^i \ ,$$

  where we have used that $\mathrm{Tr}(\sigma_i) = 0$, so only products proportional to the identity give a non-zero contribution.

[1] Notice that $x = x^0X_0 + x^iX_i$ (capital $X_0 = i\mathbf{1}, X_i = i\sigma_i$ are basis vectors, while x^0, x^i are components), but the zero component does not enter in the expression of the matrix elements of the adjoint action because it defines an Abelian ideal of the algebra.

- For the Killing product, writing the matrix elements using previous results,

$$k(x,y) = \mathrm{Tr}\left(\mathrm{ad}_x \mathrm{ad}_y\right) = \left(\mathrm{ad}_x\right)^i{}_j \left(\mathrm{ad}_y\right)^j{}_i = 4x^k y^l \epsilon_{ijk}\epsilon_{jil} = -8x^i y^i \ .$$

Notice that the product α is negative definite but non-degenerate, while k is actually degenerate due to the Abelian piece of the algebra, $k(i\mathbf{1}, y) = 0$ for any $y \in \mathfrak{u}(2)$.

Section d)

$\mathfrak{su}(2)$ is a subalgebra of $\mathfrak{u}(2)$: that formed by vectors of the form $x = x^i X_i = ix^i\sigma_i$ (so, vanishing x^0 component). In particular, the previous results can be immediately applied to get:

- For the product α, $\alpha(x,y) = -2x^i y^i$.
- For the Killing product, $k(x,y) = -8x^i y^i$.

Notice that, as it must be for a semi-simple Lie algebra like $\mathfrak{su}(2)$ (which is actually simple), the Killing form is now non-degenerate (and negative definite, as it corresponds for a compact algebra in the mathematicians convention, which is the one used when taking anti-Hermitian generators $i\sigma_j$).

6. The derived algebra

For a given Lie algebra L, take

$$L' = [L, L] \equiv \mathrm{span}\{[x,y] \,/\, x, y \in L\} \ .$$

This is by construction a vector subspace of L (we are taking all possible linear combinations of commutators). It is actually more than that, because $[x,y] \in L'$ for any $x \in L'$ and $y \in L$ (this is again by construction). Thus, L' is a subalgebra of L and an ideal. If L is a simple Lie algebra, there are no non-trivial ideals, so either $L' = 0$ (the trivial subspace) or $L' = L$. The first case is impossible because then all commutators of elements in L would vanish, making it an Abelian algebra (which is incompatible with being a a simple algebra). We conclude $L' = L$ and thus all elements of L can be written as linear combinations of commutators for a simple Lie algebra.

7. $\mathfrak{su}(2)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

Section a)

$\mathfrak{su}(2)$ is clearly non-Abelian, so let us prove that there are no ideals other than 0 (the trivial subspace) and the full $\mathfrak{su}(2)$ itself. The intuitive idea behind this is that commutation relations $[X_i, X_j] = -\frac{1}{4}[\sigma_i, \sigma_j] = -\frac{i}{2}\epsilon_{ijk}\sigma_k = \epsilon_{ijk}X_k$ mix all elements of the algebra, so they do not leave subspaces invariant (we are using conventions for the basis elements of

$\mathfrak{su}(2)$ different from the ones of question 6 because commutation relations turn out to be nicer, so $X_i = -\frac{i}{2}\sigma_i$ now). Consider a general 1-dimensional subspace, $L_1 = \text{span}\{\lambda^i X_i\}$. Taking commutators with the basis elements of the algebra and demanding them to be part of this subspace, we obtain:

$$[\lambda^i X_i, X_j] = \lambda^i \epsilon_{ijk} X_k = c_j \lambda^k X_k \ ,$$

for some $c_j \in \mathbb{R}$ (these are actually three equations, one for each j, thus the label in the constant). This condition can be rewritten as $\left(\lambda^i \epsilon_{ijk} - c_j \lambda^k\right) X_k = 0$, therefore it must be the case $\lambda^i \epsilon_{ijk} - c_j \lambda^k = 0$ for all k. These set of equations only have the trivial solution $\lambda^i = 0$ for all i. To see it, write them as

$$\left(\epsilon_{jki} - c_j \delta_{ki}\right) \lambda^i = 0 \ ,$$

which are nine equations for the three unknowns λ^i. Taking the cases $j = 1, k = 2$ and $j = 1, k = 3$ we get

$$\begin{aligned} \lambda^3 - c_1 \lambda^2 = 0 \ , \\ -c_1 \lambda^3 - \lambda^2 = 0 \ . \end{aligned}$$

This is an invertible system irrespective of the value of c_1, so $\lambda^2 = \lambda^3 = 0$. Doing the same for other j, say $j = 2$ (and considering $k = 1$ and $k = 3$) we obtain $\lambda^1 = \lambda^3 = 0$. All in all, $\lambda^i = 0$ for any i, which would mean L_1 is the trivial subspace. Thus, there are no one-dimensional ideals.

What about two-dimensional ones? Instead of trying all possibilities, we can use a trick. $\mathfrak{su}(2)$ has a non-degenerate inner product (e.g., the one given by the Killing form, which in our current basis is $k(X_i, X_j) = -2\delta_{ij}$). If we find a two-dimensional ideal, call if L_2, we can then consider the orthogonal complement $L_2^\perp$, which is one-dimensional because $L_2 \oplus L_2^\perp = \mathfrak{su}(2)$ and $\mathfrak{su}(2)$ is three-dimensional (for this it is essential the non-degeneracy of the product). Invariance of the Killing form shows that $L_2^\perp$ is an ideal. Indeed, take $v \in L_2$, $w \in L_2^\perp$ and a general $x \in \mathfrak{su}(2)$. Since L_2 is an ideal, $[v, x] \in L_2$, and then

$$0 = k([v, x], w) = k(v, [x, w]) \Rightarrow [x, w] \in L_2^\perp \ ,$$

so $[w, x] \in L_2^\perp$ for any $x \in \mathfrak{su}(2)$ and $w \in L_2^\perp$, thus $L_2^\perp$ would be a one-dimensional ideal. We showed before that these do not exist, so there cannot be two-dimensional ideals either. We conclude that $\mathfrak{su}(2)$ is simple, since there are no non-trivial ideals.

Section b)

Take now $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. We have six generators, X_i and $\tilde{X}_i$, obeying commutation relations

$$[X_i, X_j] = \epsilon_{ijk} X_k \ , \quad [\tilde{X}_i, \tilde{X}_j] = \epsilon_{ijk} \tilde{X}_k \ , \quad [X_i, \tilde{X}_j] = 0 \ .$$

This algebra is clearly not simple because there are two obvious ideals: $\mathfrak{su}(2)_1 = \text{span}\{X_i\}$ and $\mathfrak{su}(2)_2 = \text{span}\{\tilde{X}_i\}$ (names make clear that these are $\mathfrak{su}(2)$ ideals, subscripts indicate we have two of them). Are there any other ideals? Suppose we can find one such ideal, call it I. Define the linear subspaces obtained by taking commutators $I_1 = [\mathfrak{su}(2)_1, I] = \text{span}\{[x,y]\,/\,x \in \mathfrak{su}(2)_1, y \in I\}$, and similarly $I_2 = [\mathfrak{su}(2)_2, I]$. We will argue for I_1, identical statements are valid for I_2 with the obvious changes. I_1 is a subspace of I and of $\mathfrak{su}(2)_1$, because these are ideals. It is also an ideal of $\mathfrak{su}(2)_1$ actually.[2] To see this, notice that any $z \in I_1$ is an element of I, and then for any $x \in \mathfrak{su}(2)_1$ is $[z,x] = -[x,z] \in I_1$ by definition. Being $\mathfrak{su}(2)_1$ simple and I_1 an ideal, it must be either $I_1 = 0$ or $I_1 = \mathfrak{su}(2)_1$. Let us study the first case, $I_1 = 0$. In that case, if we write any element of I as $y = y_1 + y_2 \in I$ with $y_1 \in \mathfrak{su}(2)_1$ and $y_2 \in \mathfrak{su}(2)_2$ (this we can do in a unique way because the whole algebra is a direct sum), we have that for $x \in \mathfrak{su}(2)_1$, $0 = [x,y] = [x,y_1]$. There is no non-zero element in $\mathfrak{su}(2)_1$ which commutes with all the elements of that subalgebra (otherwise it would define an Abelian ideal), so $y_1 = 0$ necessarily. We then conclude that, if $I_1 = [\mathfrak{su}(2)_1, I] = 0$, then any $y \in I$ is written as $y = y_2 \in \mathfrak{su}(2)_2$, so $I \subset \mathfrak{su}(2)_2$ and being it an ideal it must be either 0 or the whole $\mathfrak{su}(2)_2$. Analogous reasoning starting from I_2 leads us to conclude that if $I_2 = 0$, either $I = \mathfrak{su}(2)_1$ or $I = 0$. The only possibility remaining is that $I_1 = \mathfrak{su}(2)_1$ and $I_2 = \mathfrak{su}(2)_2$. Then $I \subset \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$ trivially, and also $\mathfrak{su}(2)_1 = I_1 \subset I$, $\mathfrak{su}(2)_2 = I_2 \subset I$, so $I = \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$. We conclude that there are no ideals other than $0, \mathfrak{su}(2)_1, \mathfrak{su}(2)_2$ and $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$, so the algebra is semi-simple.

8. Direct sums of simple algebras

Section a)

This is just a generalization of the previous result. Let $L = \bigoplus_{i=1}^n L_i$ be a direct sum of simple Lie algebras. Each L_i is clearly an ideal of L which is non-Abelian (because they are simple). We will show that there are no ideals other than direct sums of these (which are non-Abelian) and the trivial ones (0 and the whole L), thus L will be proved to be semi-simple. For this, assume we have an ideal I. Define $I_i = [L_i, I] \equiv \text{span}\{[x,y]\,/\,x \in L_i, y \in I\}$. Since both L_i and I are ideals, $[x,y]$ is an element of L_i and I for any $x \in L_i$, $y \in I$. Thus, I_i is a linear subspace of L_i and of I. It is actually also an ideal, in particular of L_i, because for $z \in I_i$, $z \in I$ so $[z,x] \in I_i$ for $x \in L_i$. Since L_i is simple, this means I_i must be either 0 or the whole L_i. Let us write now a general element of I as $y = \sum_i y_i \in I$ with each $y_i \in L_i$ (this can be done in a unique way because the whole algebra is a direct sum of the L_i). Now, we showed that $I_j = [L_j, I]$ is either 0 or the whole L_j. If $I_j = 0$, then $[x,y] = [x,y_j] = 0$ for any $x \in L_j$. This implies $y_j = 0$, because if it were not so there would exist a non-zero element in L_j (this y_j) that commutes with all the $x \in L_j$, thus defining a one-dimensional Abelian ideal (impossible because L_j is simple). Consequently, any $y \in I$ can be written as a direct sum of elements of the L_k such that $I_k = [L_k, I] = L_k$, thus proving $I \subset \bigoplus_{k\,/\,I_k = L_k} L_k$. Since for those k $L_k = I_k \subset I$, we have the equality and $I = \bigoplus_{k\,/\,I_k = L_k} L_k$. All the ideals are then direct sums of the L_i's: $0, L_1, L_2, \ldots, L_1 \oplus L_2, L_1 \oplus L_3, \ldots, L_1 \oplus L_2 \oplus L_3, \ldots$. This proves the desired statement: L is semi-simple.

[2] Also of I, but this is not relevant for the argument.

Section b)

We know that in simple algebras we can write all elements as linear combinations of commutators (question 6). Take $x_i \in L_i$ and $x_j \in L_j$ for $j \neq i$. Write x_i as linear combination of commutators, so $x_i = \sum_a c_a[y_i^a, z_i^a]$ with $y_i^a, z_i^a \in L_i$, and now taking the product:

$$g(x_i, x_j) = g\left(\sum_a c_a[y_i^a, z_i^a], x_j\right) = \sum_a c_a g\left([y_i^a, z_i^a], x_j\right) = \sum_a c_a g\left(y_i^a, [z_i^a, x_j]\right) = 0 \ ,$$

where we used bilinearity and invariance of the product, and the fact that $[L_i, L_j] = 0$. The different factors are therefore orthogonal.

Section c)

Section b) essentially shows that there are no off-diagonal terms in the product mixing the different factors, so we can always write $g = \sum_i \tilde{g}_i$ with $\tilde{g}_i$ the restriction of g to L_i. Explicitly, for $x = \sum_i x_i$ and $y = \sum_j y_j$ with $x_k, y_k \in L_k$ the unique decomposition of general $x, y \in L$, we have

$$g(x, y) = \sum_{i,j} g(x_i, y_j) = \sum_i g(x_i, y_i) \equiv \sum_i \tilde{g}_i(x_i, y_i) \ .$$

Now, given bilinear, symmetric, invariant forms g_i in each of the L_i, if these algebras are compact in addition to simple the bilinear form is uniquely determined up to multiplicative factor. Therefore, each of the $\tilde{g}_i$, which is an induced bilinear, symmetric, invariant form on L_i, will be $\tilde{g}_i = \lambda_i g_i$ for some constants λ_i. This proves the result of the question.

9. Antisymmetry of structure constants

This is an immediate consequence of question 4 with the metric given by the trace in the question,

$$\lambda C^d{}_{ab}\delta_{dc} = \lambda C^d{}_{bc}\delta_{ad} \Leftrightarrow C_{cab} = C_{abc} = -C_{acb} \ .$$

The constants are antisymmetric in the first two indices (in the last two they are by definition), so they are totally antisymmetric.

10. Generators of simple algebras are traceless

This is again a consequence of question 6: elements of simple Lie algebras are linear combinations of commutators. Let $X = \sum_a c_a[Y^a, Z^a]$ be any such element of a simple Lie algebra, then,

$$\mathrm{Tr}(X) = \mathrm{Tr}\left(\sum_a c_a[Y^a, Z^a]\right) = \sum_a c_a \mathrm{Tr}\left([Y^a, Z^a]\right) = 0 \ ,$$

because traces of commutators are always zero by the cyclic property. The previous result is valid in any representation, since any representation of a Lie algebra must respect commutation relations.